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On Systems of Vector-Valued Linear Inequalities

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Let E be a locally convex topological vector space and L be a finite-dimensional vector lattice. We shall study systems of linear inequalities of the form

$$F(x_\nu) \geqslant a_\nu, \qquad \nu \in I,$$

where $\{x_\nu\}_{\nu \in I}$ is an indexed set of elements of E and $\{a_\nu\}_{\nu \in I}$ is a corresponding set of elements of L. We obtain a consistency condition for the system and a duality theorem for a linear extremum problem with the system as constraints.

1. INTRODUCTION

Let E be a real Hausdorff locally convex topological vector space, let E' be the dual space of E and let R be the field of real numbers. Fan [2] studied systems of linear inequalities of the form

$$f(x_\nu) \geqslant \alpha_\nu, \qquad \nu \in I, \tag{1}$$

where $\{x_\nu\}_{\nu \in I}$ is an indexed set of elements of E and $\{\alpha_\nu\}_{\nu \in I}$ is a corresponding set of real numbers. The set I of indices is of arbitrary cardinality. The system (1) is said to be *consistent*, if there exists an $f \in E'$ satisfying (1). In [2] a consistency condition for the system (1) and a duality theorem for a linear extremum problem with the system (1) as constraints are obtained in terms of the closed convex cone in $E \times R$ spanned by the set $\{(x_\nu, \alpha_\nu)\}_{\nu \in I}$. We shall deal with systems of linear inequalities of the form

$$F(x_\nu) \geqslant a_\nu, \qquad \nu \in I, \tag{1'}$$

where $\{x_\nu\}_{\nu \in I}$ is an indexed set of elements of E, $\{a_\nu\} \nu \in I$ is a

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corresponding set of elements of a finite-dimensional vector lattice L and F stands for a continuous linear mapping of E into L. We shall obtain a consistency condition for the system (1′) and a duality theorem for a linear extremum problem with the system (1′) as constraints.

2. PRELIMINARIES

The topologies appearing below are always assumed to satisfy the Hausdorff separation axiom, and the scalars of vector spaces are always real.

Let E be a locally convex topological vector space, and L be a finite-dimensional vector lattice with the dimension p, that is, L has a vector order $\geqslant$ with which L is a lattice. We assume that the vector order $\geqslant$ is *anti-symmetric*, that is, $x=y$ is equivalent to $x \geqslant y$ and $y \geqslant x$ for any elements x and y in L. We always regard L as a topological vector space with the usual Hausdorff vector topology. The set $\{a \in L: a \geqslant 0\}$ is said to be the *positive cone* of L and is denoted by K. The positive cone K is a convex cone, that is, $K+K \subset K$ and $\lambda K \subset K$ for $\lambda \geqslant 0$. We can take a Hamel basis $\{e_1,..., e_p\}$ of L such that $K=\{\sum_{i=1}^{p} \lambda_i e_i: \lambda_i \geqslant 0\}$ (cf. [1, Theorem 3.2]). Hence the positive cone K is closed and has a nonempty interior int K. For two elements x and y of L, we write $x > y$ when $x-y$ belongs to int K. Moreover L has the *least upper bound property*, that is, any nonempty subset of L with an upper bound has a least upper bound. If a subset A of L has a least upper bound, then it is unique. So, we denote by $\sup A$ the least upper bound. Similarly, we denote by $\inf A$ the greatest lower bound of A. The dual basis of $\{e_1,..., e_p\}$ is denoted by $\{e'_1,..., e'_p\}$, that is, e'_i is a linear functional on L such that $e'_i(e_j)=\delta_{ij}$ for $i, j=1,..., p$, where δ_{ij} is the Kronecker's delta. We note that if a subset A of L has an upper bound, then we have

$$\sup A = \sum_{i=1}^{p} (\sup e'_i(A))\, e_i,$$

and similarly we have

$$\inf A = \sum_{i=1}^{p} (\inf e'_i(A))\, e_i.$$

We denote by $\mathbf{L}(E, L)$ the space of all continuous linear mappings of E into L. We consider systems of linear inequalities of the form

$$F(x_\nu) \geqslant a_\nu, \qquad \nu \in I, \tag{2}$$

where $\{x_\nu\}_{\nu \in I}$ is an indexed set of elements of E and $\{a_\nu\}_{\nu \in I}$ is a corresponding set of elements of L. We denote by $\mathbf{S}$ the set of all F in $\mathbf{L}(E, L)$ satisfying (2). The system (2) is said to be *consistent* if the set $\mathbf{S}$ is nonempty. An element of $\mathbf{S}$ is said to be a *solution* for the system (2). We denote by C the closed convex cone in $E \times L$ spanned by the set $\{(x_\nu, a_\nu)\}_{\nu \in I}$, and by Cx the set $\{a \in L\colon (x, a) \in C\}$ for each x in E, that is, the section of C at x. A set $D(C) = \{x \in E\colon Cx \text{ is not empty}\}$ is said to be the *domain* of C.

3. RESULTS

THEOREM 1. *The statement* (i) *implies* (ii) *and the statement* (ii) *implies* (iii):

- (i) *The set* $C0$ *is contained in* $-\operatorname{int} K \cup \{0\}$;
- (ii) *the system* (2) *is consistent*;
- (iii) *the set* $C0$ *is contained in* $-K$.

Proof. (i) implies (ii). If $Tx = Cx \backslash (-\operatorname{int} K)$ for each x in E, then it follows that for any neighborhood V of 0 in L, there exists a neighborhood U of 0 in E such that $Tx \subset V$ for all x in U. In fact, if the assertion is false, then there exists an open neighborhood V_0 of 0 in L such that the closure of V_0 is compact, and for any neighborhood U of 0 in E, there exists an element x_U in U such that $Tx_U \not\subset V_0$. Let a_U be an element of $Tx_U \backslash V_0$. For each a_U, there exists a number λ_U such that $0 < \lambda_U \leqslant 1$ and $\lambda_U a_U$ belongs to the boundary ∂V_0 of V_0. Let $y_U = \lambda_U x_U$ and $b_U = \lambda_U a_U$. We consider nets $\{y_U\}$ and $\{b_U\}$ indexed by the neighborhood system of 0 in E. Since ∂V_0 is compact, considering a subnet, we can assume that the net $\{b_U\}$ converges to an element b_0 of ∂V_0. Since b_U belongs to neither V_0 nor $-\operatorname{int} K$, so do b_0. On the other hand, the net $\{y_U\}$ converges to 0. Since the convex cone C is closed, $(0, b_0)$ belongs to C, that is, $b_0 \in C0$. This contradicts the statement (i). Let e be an element of $\operatorname{int} K$. Then $e - K$ is a neighborhood of 0, and hence there exists a convex symmetric open neighborhood W of 0 in E such that $Tx \subset e - K$ for all x in W. Let $A = W \times \{e\}$. If we denote by P_E the projection of $E \times L$ onto E, then we have

$$0 \in W = \operatorname{int} P_E(A) \subset \operatorname{int} P_E(A - C).$$

Let $(x, e) \in A$ and $(x, a) \in C$. If a does not belong to $-\operatorname{int} K$, then a belongs to Tx, and hence $a \in e - K$, that is, $a \leqslant e$. If a belongs to $-\operatorname{int} K$, then obviously $a \leqslant e$. Therefore, from [4, Theorem 1], there exists a linear mapping F of E into L such that

$$F(x) - a \geqslant F(y) - e \qquad \text{for } (x, a) \in C \text{ and } y \in W. \tag{3}$$

From the case $y = 0$, it follows that

$$F(x) - a + e \geqslant 0 \qquad \text{for} \quad (x, a) \in C.$$

Since C is a convex cone, we have

$$F(x) - a + e/\lambda \geqslant 0 \qquad \text{for} \quad \lambda > 0.$$

Since the positive cone K is closed, we have

$$F(x) - a \geqslant 0 \qquad \text{for} \quad (x, a) \in C.$$

In particular, we have

$$F(x_\nu) \geqslant a_\nu \qquad \text{for} \quad \nu \in I.$$

From the case $(x, a) = 0$ in (3), we have

$$e \geqslant F(y) \qquad \text{for} \quad y \in W.$$

Since W is symmetric, we have

$$e \geqslant F(y) \geqslant -e \qquad \text{for} \quad y \in W.$$

Since the interval $[-e, e]$ is bounded in the sense of the vector topology, we can conclude that the linear mapping F is continuous.

(ii) implies (iii). From the statement (ii), there exists an element F of $\mathbf{L}(E, L)$ such that

$$F(x_\nu) - a_\nu \geqslant 0 \qquad \text{for} \quad \nu \in I.$$

If we define a linear mapping ψ of $E \times L$ into L by

$$\psi(x, a) = F(x) - a,$$

then ψ is continuous. Since C is the closed convex cone spanned by the set $\{(x_\nu, a_\nu)\}_{\nu \in I}$ and the positive cone K is closed, it follows that

$$\psi(x, a) \geqslant 0 \qquad \text{for} \quad (x, a) \in C.$$

If a is an element of $C0$, then we have

$$0 \leqslant \psi(0, a) = -a,$$

that is, a belongs to $-K$.

Remark 1. It is easily seen that the statements (i) and (ii) are not equivalent when the domension of the vector lattice L is more than one, but it is not known whether the statements (ii) and (iii) are equivalent in general.

We proceed to study a linear extremum problem with the system (2) as constraints.

LEMMA 1. *Let X be a finite-dimensional topological vector space and Y be a topological vector space and T be a continuous linear mapping of X into Y. If C is a subset of X such that $C + \ker T$ is closed, then the image $T(C)$ of C under T is closed, where* $\ker T$ *denotes the kernel of T.*

Proof. Since the vector topology of X can be described by an inner product $\langle \cdot, \cdot \rangle$, we can regard X as a Hilbert space. Suppose that y is an element of the closure of $T(C)$, and $\{x_n\}$ is a sequence in C such that a sequence $\{Tx_n\}$ converges to y. Let P be the orthogonal projection of X onto $\ker T$ and $d_n = |x_n - Px_n|$, where $|\cdot|$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle$. If the sequence $\{d_n\}$ is unbounded, then we can assume that $d_n > n$ by considering a subsequence. If $u_n = (x_n - Px_n)/d_n$, then $|u_n| = 1$, and hence we can assume that the sequence $\{u_n\}$ converges to an element u_0 of X. Then

$$\begin{aligned} Tu_0 &= \lim_n Tu_n = \lim_n (Tx_n - TPx_n)/d_n \\ &= \lim_n Tx_n/d_n = 0. \end{aligned}$$

Since $Px_n + d_n u_0$ belongs to $\ker T$, we have

$$|x_n - Px_n - d_n u_0| \geqslant d_n,$$

that is, $|u_n - u_0| \geqslant 1$, which is a contradiction. Hence the sequence $\{d_n\}$ is bounded. If $v_n = x_n - Px_n$, then we can assume that the sequence $\{v_n\}$ converges to an element v of X. Since $C + \ker T$ is closed, v belongs to $C + \ker T$, and hence we can write $v = v' + v''$ with $v' \in C$ and $v'' \in \ker T$. Then we have

$$Tv' = Tv = \lim_n Tv_n = \lim_n Tx_n = y.$$

Hence y belongs to $T(C)$.

THEOREM 2. *Let the topological vector space E be finite dimensional and the set $C0$ be contained in $-\operatorname{int} K \cup \{0\}$. Let y be an element of E. Then the set $\{F(y) : F \in \mathbf{S}\}$ is bounded below if and only if y belongs to $D(C)$. Furthermore, when this condition is satisfied, the equation*

$$\inf\{F(y) : F \in \mathbf{S}\} = \sup Cy$$

holds.

Proof. If there exists an element b of L such that $(y, b) \in C$, then $F(y) \geqslant b$ for all F in $\mathbf{S}$, since C is a closed convex cone spanned by $\{(x_\nu, a_\nu)\}_{\nu \in I}$ and the positive cone K is closed. Hence $\{F(y) : F \in \mathbf{S}\}$ has a lower bound b. Conversely let the set $\{F(y) : F \in \mathbf{S}\}$ be bounded below. We consider, for a fixed i, a system of real-valued linear inequalities of the form

$$f(x_\nu) \geqslant e_i'(a_\nu), \qquad \nu \in I. \tag{4}$$

From Theorem 1 the system (2) has a solution F, and the composite $e_i' \circ F$ of e_i' and F is a solution of (4). For any solution f for (4), we define an element G of $\mathbf{L}(E, L)$ by

$$Gx = f(x) e_i + \sum_{j \neq i} (e_j' \circ F)(x) e_j \qquad \text{for} \quad x \in E.$$

Then G is a solution for the system (2). If b is a lower bound of the set $\{F(y) : F \in \mathbf{S}\}$, then

$$b \leqslant G(y) = f(y) e_i + \sum_{j \neq i} (e_j' \circ F)(y) e_j,$$

and hence $e_i'(b) \leqslant f(y)$. Hence $e_i'(b)$ is a lower bound of the set $\{f(y) : f \in \mathbf{S}_i\}$, where $\mathbf{S}_i$ is the set of all solutions for the system (4). From [2, Theorem 3], there exists a real number β such that $(y, \beta) \in C_i$, the closed convex cone spanned by $\{(x_\nu, e_i'(a_\nu))\}_{\nu \in I}$. Let T_i be a continuous linear mapping of $E \times L$ into $E \times R$ defined by

$$T_i(x, a) = (x, e_i'(a)).$$

If $(x, a) \in \ker T_i \cap C$, then $x = 0$, $e_i'(a) = 0$ and $a \in C0$. From the hypothesis a belongs to $-\operatorname{int} K \cup \{0\}$. Since $e_i'(a) = 0$, a must be 0. Hence $\ker T_i + C$ is closed by [3, (2.1)], and hence $T_i(C)$ is closed by Lemma 1. Since $(x_\nu, e_i'(a_\nu))$ belongs to $T_i(C)$ is closed by Lemma 1. Since $(x_\nu, e_i'(a_\nu))$ belongs to $T_i(C)$ for every $\nu \in I$, we have $C_i \subset T_i(C)$. Hence $(y, \beta) \in T_i(C)$, that is, there exists an element (x, b) of C such that $x = y$, $e_i'(b) = \beta$. Hence Cy is not empty.

Let y be an element of $D(C)$. Then, it is obvious that $\inf\{F(y) : F \in \mathbf{S}\} \geqslant$

$\sup Cy$. We show the reverse ineqality. Let $c > \sup Cy$ and let b be an element of L such that $c > b > \sup Cy$. Then for $i = 1, \ldots, p$, we have $e_i'(b) > \sup e_i'(Cy)$. Since $e_i'(Cy) = T_i(C)y$, we have $e_i'(b) > \sup T_i(C)y$, where $T_i(C)y$ is the section of the convex cone $T_i(C)$ in $E \times R$ at y. Hence $(y, e_i'(b))$ does not belong to $T_i(C)$, which is closed. So, there exist an f in E and a real number λ such that

$$f(y) + \lambda e_i'(b) > f(x) + \lambda\mu \qquad \text{for} \quad (x, \mu) \in T_i(C).$$

From the case $x = y$, we have

$$\lambda e_i'(b) > \lambda\mu \qquad \text{for} \quad \mu \in T_i(C)y.$$

From the inequality we can conclude that $\lambda > 0$. If $g = f/\lambda$, then

$$g(y) + e_i'(b) > g(x) + \mu \qquad \text{for} \quad (x, \mu) \in T_i(C).$$

By the continuity of g, there exists a convex symmetric open neighborhood W_i of 0 such that

$$g(y) - g(x) < e_i'(c) - e_i'(b) \qquad \text{for all} \quad x \in y + W_i.$$

Hence we have if $x \in (y + W_i) \cap D(C)$, then $\mu < e_i'(c)$ for any $\mu \in T_i(C)x$. If $W = \bigcap_{i=1}^p W_i$, then W is a convex symmetric open neighborhood of 0 such that if $x \in (y + W) \cap D(C)$, then $a < c$ for $a \in Cx$. If $A = (y + W) \times \{c\}$, then we have $a \leqslant c$ for any $(x, c) \in A$ and $(x, a) \in C$. By [4, Theorem 1], there exists a linear mapping F of E into L such that

$$F(x) - c \leqslant F(z) - a \qquad \text{for } x \in y + W \text{ and } (z, a) \in C.$$

From the case $(z, a) = 0$ and $x = y$, we have $F(y) \leqslant c$. By the same argument in the proof of Theorem 1, we can conclude that F is a solution for (2). Hence we have proved that for any c with $c > \sup Cy$, there exists an F in S such that $c \geqslant F(y)$. Hence we have $\inf\{F(y): F \in \mathrm{S}\} \leqslant \sup Cy$.

Remark 2. In Theorem 2 the "sup" of $\sup Cy$ cannot be replaced by "max" when the dimension of L is more than one. The following simple example is illuminating: Let $E = R$, $L = R^2$ and $K = \{(x^1, x^2): x^1 \geqslant 0$ and $x^2 \geqslant 0\}$. A system of linear inequalities is

$$F(1) \geqslant (-1, 0),$$
$$F(1) \geqslant (0, -1).$$

In this case, $C0 = \{(0, 0)\}$ and the hypothesis of Theorem 2 is satisfied. On the other hand, $C1 = \{-(\lambda, \mu): \lambda + \mu = 1, \ \lambda \geqslant 0, \ \mu \geqslant 0\}$, and hence $\sup C1 = (0, 0) \notin C1$.

Remark 3. Finite dimensionality of the topological vector space E is assumed in Theorem 2. We do not know whether the assumption may be removed. But for a finite system

$$F(x_i) \geqslant a_i, \qquad 1 \leqslant i \leqslant n, \tag{5}$$

where $x_i \in E$ and $a_i \in L$, we can remove the assumption, for the convex cone $T_i(C)$ appearing in the proof of Theorem 2 is a polyhedral cone and closed without the finite dimensionality of E. Moreover we can replace the assumption that the set $C0$ is contained in $-\operatorname{int} K \cup \{0\}$ by the weaker assumption that the finite system (5) is consistent. Hence we have the following corresponding theorems for the finite system (5).

THEOREM 1′. *The statement* (i) *implies* (ii) *and the statement* (ii) *implies* (iii) *for the finite system* (5):

(i) *For any* n *nonnegative numbers* λ_i $(1 \leqslant i \leqslant n)$, *the relation* $\sum_{i=1}^n \lambda_i x_i = 0$ *implies* $\sum_{i=1}^n \lambda_i a_i = 0$ *or* $\sum_{i=1}^n \lambda_i a_i < 0$;

(ii) *the finite system* (5) *is consistent*;

(iii) *for any* n *nonnegative numbers* λ_i $(1 \leqslant i \leqslant n)$, *the relation* $\sum_{i=1}^n \lambda_i x_i = 0$ *implies* $\sum_{i=1}^n \lambda_i a_i \leqslant 0$.

THEOREM 2′. *Let the finite system* (5) *be consistent, and let* $\mathbf{S}$ *be the set of all solutions for the finite system* (5). *Let* y *be an element of* E. *Then the set* $\{F(y) \colon F \in \mathbf{S}\}$ *is bounded below if and only if* y *is a linear combination of* $x_1, \ldots, x_n$ *with nonnegative coefficients. Furthermore, when this condition is satisfied, the equation*

$$\inf\{F(y) \colon F \in \mathbf{S}\} = \sup\left\{\sum_{i=1}^n \lambda_i a_i \colon \lambda_i \geqslant 0 \ (1 \leqslant i \leqslant n),\ y = \sum_{i=1}^n \lambda_i x_i\right\}$$

holds.

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